Lecture 2 Supplement: Bumps

Let $h(x) = \begin{cases} e^{-1/1-x^2} & |x|<1 \\ 0 & |x| \geq 1 \end{cases}$

$h'(x) = 0$ if $|x|>1$

$h'(x) = e^{-1/1-x^2}\left(\frac{1}{(1-x^2)^2}\right)(-2x)$ if $|x|<1$

$$\lim_{t\to 0^-} \frac{h(1+t)-h(1)}{t} = \lim_{t\to 0^-} \frac{\exp\left(-\frac{1}{1-(1+t)^2}\right)}{t} = \lim_{t\to 0} \frac{1/t}{\exp\left(\frac{1}{1-(1+t)^2}\right)}$$

L'Hopital $\to$ $= \lim_{t\to 0^-} \dfrac{-1/t^2}{\exp\left(\frac{1}{1-(1+t)^2}\right)\left(\frac{2(1+t)}{(1-(1+t)^2)^2}\right)}$

$= \lim_{t\to 0^-} \dfrac{-(1-(1+t)^2)^2}{2t^2(1+t)} \exp\left(\frac{-1}{1-(1+t)^2}\right) = 0.$ A similar argument holds at -1

So $h'(-1) = h'(1) = 0$ and

$$h'(x) = \begin{cases} \frac{-2x}{(1-x^2)^2}\exp(-1/1-x^2) & |x|<1 \\ 0 & |x| \geq 1 \end{cases}$$

We posit that $h^{(m)}(x) = \begin{cases} \frac{q_m(x)}{(1-x^2)^{m+1}}\exp(-1/1-x^2) & |x|<1 \\ 0 & |x|\geq 1 \end{cases}$

for a polynomial $q_m(x)$. Indeed, this may be proved by L'Hopital's rule as above. Thus, $h(x) \in C_c^\infty(\mathbb{R})$

- Define $\tilde{\varphi}(x) = \int_{-\infty}^{x} h(s)\,ds$, which is finite since $e^{-\frac{1}{1-x^2}} \leq 1/e$.

  Then, let $a = \int_{-\infty}^{\infty} h(s)\,ds$ and $\frac{1}{a}\tilde{\varphi}(x) := \varphi(x)$

  has $\varphi(x) = 0$ for $x \leq -1$, $\varphi(x) = 1$ for $x \geq 1$, and

  $\varphi \in C^\infty(\mathbb{R})$. This is called the "transition function". By

"Less Important for us"

- Let $A \subseteq \mathbb{R}^n$ be compact ~~and $A \subseteq U$ open~~.

Let $d(A,x) = \inf_{y\in A} d(x,y)$.

$\varphi(1 - \frac{1}{\varepsilon} d(A,x)^2) = \psi_A(x)$ gives a "smooth bump"

So $\psi_A = 1$ on A and $\psi_A(x) = 0$ if $d(A,x) > \sqrt{2\varepsilon}$